*To make a million, start with $900,000.*
**Morton Shulman,** Politician, Businessman, and Television Personality

# Compound Interest Formula 3-5

## Key Terms

- compound interest formula
- annual percentage rate (APR)
- annual percentage yield (APY)

## Objectives

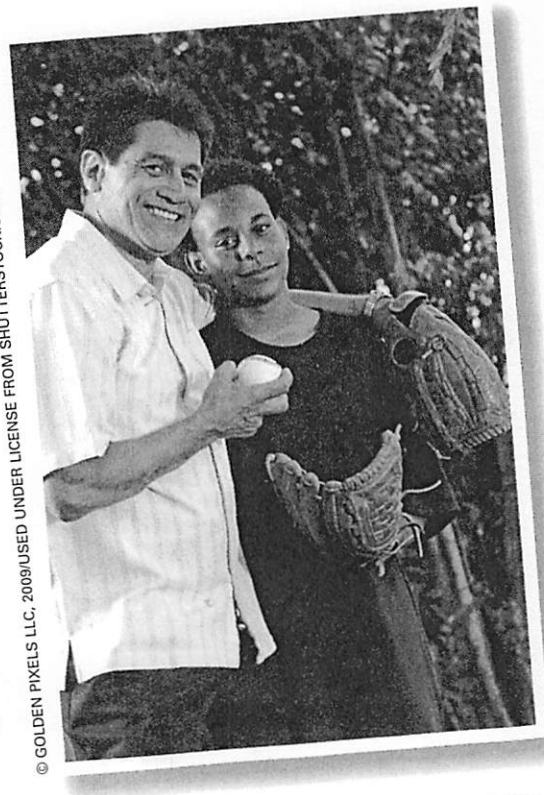
- Become familiar with the derivation of the compound interest formula.
- Make computations using the compound interest formula.

## What are the advantages of using the compound interest formula?

Julio deposited $10,000 in a five-year CD, with the intention of using the money for his son's college education. The account pays 5.2% interest compounded daily. There will be no deposits or withdrawals during the five years. Julio wants to know how much the $10,000 will grow to by the end of the five years. Imagine if he set up a daily compound interest table as in the last lesson. There are over 1,800 days in five years, so the table would get quite tedious. It is not practical to solve this problem one day at a time.

Calculating compound interest using the simple interest formula is tedious when there are numerous periods. The power of mathematics can turn this long procedure into a relatively small amount of work. Numerical examples and algebra can be combined to uncover a pattern that leads to a formula that finds compound interest. The **compound interest formula** relates principal, interest rate, the number of times interest is compounded per year, and the number of years the money will be on deposit, and the ending balance. The formula is used for any type of compounding: annually, semiannually, monthly, weekly, daily, and so on.

In Lesson 3-3, you used the annual interest rate to compute interest. Banks call this the **annual percentage rate (APR)**. Most banks advertise the **annual percentage yield (APY)** since it is higher than the APR for accounts compounded more than once per year. The bank takes the dollar amount of interest you earn under the compounding to create the APY. The APY is the simple interest rate that would be required to give the same dollar amount of interest that the compounding gave. Therefore, annual percentage yield (APY) is an annual rate of interest that takes into account the effect of compounding.



## Skills and Strategies

Here you will solve some compound interest problems and then look for a pattern to derive the compound interest formula.

**EXAMPLE 1**

Jose opens a savings account with principal $P$ dollars that pays 5% interest, compounded quarterly. What will his ending balance be after one year?

**SOLUTION 1** Find the first quarter's interest, where $p = P$, $r = 0.05$, and $t = \frac{1}{4}$.

Use the simple interest formula. $I = prt$

Substitute. $I = (P)(0.05)\left(\frac{1}{4}\right)$

Simplify. $I = \frac{0.05}{4}P$

Let $B_1$ represent the first quarter's ending balance, the sum of $P$ and the first quarter's interest.

Principal + Interest $B_1 = P + \frac{0.05}{4}P$

Factor out $P$. $B_1 = P\left(1 + \frac{0.05}{4}\right)$

To get the second quarter's ending balance, follow the same procedure with the new balance $B_1$.

Principal + Interest $B_2 = B_1 + \frac{0.05}{4}B_1$

Factor out $B_1$. $B_2 = B_1\left(1 + \frac{0.05}{4}\right)$

Substitute $P\left(1 + \frac{0.05}{4}\right)$ for $B_1$. $B_2 = p\left(1 + \frac{0.05}{4}\right)\left(1 + \frac{0.05}{4}\right)$

Write in exponential form. $B_2 = P\left(1 + \frac{0.05}{4}\right)^2$

To get the third quarter's ending balance, follow the same procedure with the new balance $B_2$.

Principal + Interest $B_3 = B_2 + \frac{0.05}{4}B_2$

Factor out $B_2$. $B_3 = B_2\left(1 + \frac{0.05}{4}\right)$

Substitute $P\left(1 + \frac{0.05}{4}\right)^2$ for $B_2$. $B_3 = P\left(1 + \frac{0.05}{4}\right)^2\left(1 + \frac{0.05}{4}\right)$

Write in exponential form. $B_3 = P\left(1 + \frac{0.05}{4}\right)^3$

To get the fourth quarter's ending balance, follow the same procedure with the new balance $B_3$.

Factor out $B_4$. $$B_4 = B_3 + \frac{0.05}{4} B_3 = B_3\left(1 + \frac{0.05}{4}\right)$$

Substitute $P\left(1 + \frac{0.05}{4}\right)^3$ for $B_3$. $$B_4 = P\left(1 + \frac{0.05}{4}\right)^3\left(1 + \frac{0.05}{4}\right)$$

Ending balance after one year $$B_4 = P\left(1 + \frac{0.05}{4}\right)^4$$

This is the balance after one year. Examine the formula for patterns.

**CHECK YOUR UNDERSTANDING**

Rico deposits \$800 at 3.87% interest, compounded quarterly. What is his ending balance after one year? Round to the nearest cent.

**EXAMPLE 2**

If you deposit $P$ dollars for one year at 5% compounded daily, express the ending balance algebraically.

**SOLUTION** Use the formula from Example 1 and make adjustments for daily compounding. When the interest was compounded quarterly, there was a denominator of 4 and an exponent of 4 in the formula.

$$B_4 = P\left(1 + \frac{0.05}{4}\right)^4$$

With daily compounding, these entries are replaced with 365. Rewrite the formula.

Ending balance after one year $$B = P\left(1 + \frac{0.05}{365}\right)^{365}$$

This is the ending balance expressed algebraically.

**CHECK YOUR UNDERSTANDING**

Nancy deposits \$1,200 into an account that pays 3% interest, compounded monthly. What is her ending balance after one year? Round to the nearest cent.

**EXTEND YOUR UNDERSTANDING**

Nancy receives two offers in the mail from other banks. One is an account that pays 2.78% compounded daily. The other account pays 3.25% compounded quarterly. Would either of these accounts provide Nancy with a better return than her current account? If so, which account?

## Compound Interest Formula

Examples 1 and 2 involved accounts for one year. The exponent and the denominator in those formulas are the number of times the interest is compounded *in one year*. You can leave your money in for more than one year. The formula used to compute the ending balance includes the variable $t$, where $t$ is the number of years.

**Compound Interest Formula**

$$B = p\left(1 + \frac{r}{n}\right)^{nt}$$

where $B$ = ending balance
$p$ = principal or original balance
$r$ = interest rate expressed as a decimal
$n$ = number of times interest is compounded annually
$t$ = number of years

**EXAMPLE 3**

Marie deposits $1,650 for three years at 3% interest, compounded daily. What is her ending balance?

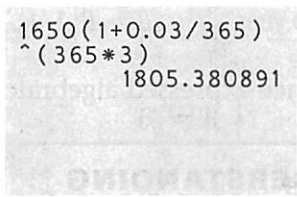
**SOLUTION** Use the compound interest formula. The values for the variables are $p = 1{,}650$, $r = 0.03$, $n = 365$, and $t = 3$.

Substitute the values for Marie's account.

$$B = 1{,}650\left(1 + \frac{0.03}{365}\right)^{365(3)}$$

Use your calculator to enter the expression. Enter the entire expression; try not to do it in separate terms. The keystrokes are:

1650(1+0.03/365)^(365×3) ENTER




Marie's ending balance, to the nearest cent, is $1,805.38.

**■ CHECK YOUR UNDERSTANDING**

Kate deposits $2,350 in an account that earns interest at a rate of 3.1%, compounded monthly. What is her ending balance after five years? Round to the nearest cent.

**■ EXTEND YOUR UNDERSTANDING**

Write an algebraic expression for the ending balance after $k$ years of an account that starts with a balance of $2,000 and earns interest at a rate of 3.5%, compounded daily.

## EXAMPLE 4

Sharon deposits $8,000 in a one year CD at 3.2% interest, compounded daily. What is Sharon's annual percentage yield (APY) to the nearest hundredth of a percent?

**SOLUTION** Find the APY using the compound interest formula and the simple interest formula.

Use the compound interest formula. $B = p\left(1 + \frac{r}{n}\right)^{nt}$

Substitute. $B = 8{,}000\left(1 + \frac{0.032}{365}\right)^{365\times 1}$

Simplify. $B = 8{,}260.13$

Subtract the principal from the new balance.

$$I = 8{,}260.13 - 8{,}000 = 260.13$$

Use the simple interest formula. $I = prt$

Solve for $r$. $r = \frac{I}{pt}$

Substitute. $r = \frac{260.13}{8{,}000 \times 1}$

Simplify. $r \approx 0.0325 = 3.25\%$

The annual percentage yield is 3.25%.

APY can also be found by using the formula $APY = \left(1 + \frac{r}{n}\right)^n - 1$, where $r$ is the interest rate and $n$ is the number of times interest is compounded per year.

Use the *APY* formula. $APY = \left(1 + \frac{r}{n}\right)^n - 1$

Substitute. $APY = \left(1 + \frac{0.032}{365}\right)^{365} - 1$

Simplify. $APY \approx 0.0325 = 3.25\%$

The annual percentage yield is 3.25%, which is the same as the previous answer.

### CHECK YOUR UNDERSTANDING

Barbara deposits $3,000 in a one year CD at 4.1% interest, compounded daily. What is the APY to the nearest hundredth of a percent?

### EXTEND YOUR UNDERSTANDING

Consider an amount $x$ deposited into a CD at 2.4% interest compounded daily, and the same amount deposited into a CD at the same rate that compounds monthly. Explain why, after 1 year, the balance on a CD that compounds daily is greater than the CD that compounded monthly.

## Applications

*To make a million, start with $900,000.*
**Morton Shulman**, Politician, Businessman, and Television Personality

1. How might these words apply to what is in this lesson?
2. Jimmy invests $4,000 in an account that pays 5% annual interest, compounded semiannually. What is his balance, to the nearest cent, at the end of 10 years?
3. On Olga's 16th birthday, her uncle invested $2,000 in an account that was locked into a 4.75% interest rate, compounded monthly. How much will Olga have in the account when she turns 18? Round to the nearest cent.
4. Samantha deposits $1,500 into the Park Street Bank. The account pays 4.12% annual interest, compounded daily. To the nearest cent, how much is in the account at the end of three non-leap years?
5. Joanne deposits $4,300 into a one-year CD at a rate of 4.3%, compounded daily.
   **a.** What is her ending balance after the year?
   **b.** How much interest does she earn?
   **c.** What is her annual percentage yield to the nearest hundredth of a percent?
6. Mike deposits $5,000 in a three-year CD account that yields 3.5% interest, compounded weekly. What is his ending balance at the end of three years?
7. Rob deposits $1,000 in a savings account at New York State Bank that pays 4.4% interest, compounded monthly.
   **a.** How much is in his account at the end of one year?
   **b.** What is the APY for this account to the nearest hundredth of a percent?
8. How much more does $1,000 earn in eight years, compounded daily at 5%, than $1,000 over eight years at 5%, compounded semiannually?
9. If $3,000 is invested at an interest rate of 4.8%, compounded hourly for two years, what is the ending balance?
10. Mike and Julie receive $20,000 in gifts from friends and relatives for their wedding. They deposit the money into an account that pays 4.75% interest, compounded daily.
    **a.** Will their money double in fourteen years?
    **b.** Will their money double in fifteen years?
11. Lindsay invests $80 in an account that pays 5% annual interest, compounded monthly. Michele invests $60 in an account that pays 8% annual interest, compounded weekly.
    **a.** Whose balance is greater after one year?
    **b.** Whose balance is greater after twelve years?

**12.** Investigate the difference between compounding annually and simple interest for parts a–j.

- **a.** Find the simple interest for a one-year CD for \$5,000 at a 6% interest rate.
- **b.** Find the interest for a one-year CD for \$5,000 at an interest rate of 6%, compounded annually.
- **c.** Compare the results from parts a and b.
- **d.** Find the simple interest for a three-year CD for \$5,000 at an interest rate of 6%.
- **e.** Find the interest for a three-year CD for \$5,000 at an interest rate of 6%, compounded annually.
- **f.** Compare the results from parts d and e.
- **g.** Find the simple interest for a six-year CD for \$5,000 at an interest rate of 4%.
- **h.** Find the interest for a six-year CD for \$5,000 at an interest rate of 4%, compounded annually.
- **i.** Compare the results from parts g and h.
- **j.** Is interest compounded annually the same as simple interest? Explain.

**13.** Rodney invests a sum of money, $P$, into an account that earns interest at a rate of $r$, compounded yearly. Gerald invests half that amount into an account that pays twice Rodney's interest rate. Which of the accounts will have the higher ending balance after one year? Explain.

**14.** Island Bank is advertising a special 6.55% APR for CDs. Manny takes out a one-year CD for \$40,000. The interest is compounded daily. Find the annual percentage yield for Manny's account to the nearest hundredth of a percent.

**15.** Businesses deposit large sums of money into bank accounts. Imagine an account with 10 million dollars in it.

- **a.** How much would the account earn in one year of simple interest at a rate of 5.12%?
- **b.** How much would the account earn in one year at 5.12% if the interest was compounded daily?
- **c.** How much more interest is earned by interest compounded daily compared to simple interest?

**16.** An elite private college receives large donations from successful alumni. The account that holds these donations has \$955,000,000 currently.

- **a.** How much would the account earn in one year of simple interest at a rate of 5.33%?
- **b.** How much would the account earn in one year at 5.33% if the interest was compounded daily? Round to the nearest cent.
- **c.** How much more interest is earned by compounded daily as compared to simple interest?
- **d.** If the money is used to pay full scholarships, and the price of tuition is \$61,000 per year to attend, how many more students can receive full four-year scholarships if the interest was compounded daily rather than using simple interest?

## Applications

*The infinite! No other question has ever moved so profoundly the spirit of man.*

David Hilbert, Mathematician

1. How might these words apply to this lesson?

2. A bank representative studies compound interest, so she can better serve customers. She analyzes what happens when \$2,000 earns interest several different ways at a rate of 4% for 3 years.
   **a.** Find the interest if it is computed using simple interest.
   **b.** Find the interest if it is compounded annually.
   **c.** Find the interest if it is compounded semiannually.
   **d.** Find the interest if it is compounded quarterly.
   **e.** Find the interest if it is compounded monthly.
   **f.** Find the interest if it is compounded daily.
   **g.** Find the interest if it is compounded hourly.
   **h.** Find the interest if it is compounded every minute.
   **i.** Find the interest if it is compounded continuously.
   **j.** What is the difference in interest between simple interest and interest compounded continuously?

3. Ed computes the ending balance for an account he is considering. The principal is \$20,000, and the interest rate is 5.39%, compounded continuously for four years. He uses the formula $B = pe^{rt}$ and substitutes directly on his calculator. Look at the keystrokes he entered.

   20,000$e$^(.0539)(4)

   He presses ENTER and sees this display.

   20000$e$^(.0539)(4) = 84430.32472

   Ed's knowledge of compound interest leads him to believe that this answer is extremely unreasonable. To turn \$20,000 into over \$84,000 in just four years at 5.39% interest seems incorrect to him.
   **a.** Find the correct ending balance.
   **b.** Explain what part of Ed's keystroke sequence is incorrect.

4. Find the interest earned on a \$50,000 deposited for six years at $4\frac{1}{8}\%$ interest, compounded continuously.

5. Whitney deposits \$9,000 for two years. She compares two different banks. State Bank will pay her 4.1% interest, compounded monthly. Kings Savings will pay her 4.01% interest, compounded continuously.
   **a.** How much interest does State Bank pay?
   **b.** How much interest does Kings Savings pay?
   **c.** Which bank pays higher interest? How much higher?
   **d.** What other factors might affect Whitney's choice besides interest?